AIAA JOURNAL
Vol. 32, No. 12, December 1994

Analytical Study of the Neutral Stability of a Model Hypersonic Boundary Layer

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The neutral modes of a hypersonic boundary layer flow over an adiabatic flat plate are considered. A formulation of the governing second-order linear equation for the pressure disturbance is developed that lends itself to the application of the WKB method over the entire boundary layer. This formulation provides analytic eigenvalues and eigenfunction relations for the pressure disturbances and is applicable to flows at moderate Mach numbers as well. Solutions are determined for the cases of the wave speed $c = 0$ and $c = 1$, and show good qualitative agreement with numerical computations as well as results in the limit $M_\infty \to \infty$.

I. Introduction

THE stability of hypersonic boundary layers has always been a complex and difficult subject. In recent years, because researchers are motivated by the potential application to the design of high-speed vehicles, much effort and thus much progress have been achieved towards the understanding of the onset and growth of instabilities in hypersonic boundary layers. The basic analysis for compressible stability theory was first developed by Lees and Lin;[1] however, most of our current understanding is due to the outstanding works of Mack,[2–4] who provided numerical computations for the inviscid and viscous stability problem and found the inviscid disturbances to be the dominant ones at higher Mach numbers. These inviscid modes exist for a wide range of wave angles. However, Mack found the inviscid, two-dimensional wave to be the most unstable; thus to examine the dominant modes, one can reduce the linear stability equations in the limit $Re \to \infty$ to the compressible Rayleigh equation. This equation was analyzed by Mack,[3] who obtained a full numerical solution. In the limit $M_\infty \to \infty$, Cowley and Hall[5] and Smith and Brown[6] examined the equation and developed a solution for the acoustic modes for a fluid satisfying the Chapman viscosity relation. In addition, Smith and Brown provided an asymptotic solution for the vorticity mode in the limit $M_\infty \to \infty$ by means of the triple deck method. Using a similar technique, Balsa and Goldstein[7] analyzed supersonic mixing layers, and Grubin and Trigub[8] examined the problem for fluids with a Prandtl number other than unity and provided calculations for the vorticity mode in boundary layer flows of helium. More recently, Blackaby et al.[9] studied the vorticity mode in the limit $M_\infty \to \infty$ for fluids with the viscosity specified by Sutherland's law. They also examined this mode in the leading-edge interaction zone.

Because of the complexity of the problem, even in linear theory, many of the results rely heavily on numerical computations and/or asymptotic methods. It is therefore of interest to develop a simplified model for this problem that can give more analytical details for a wide range of freestream Mach numbers. In the present study, for simplicity, we concentrate on the two-dimensional disturbance and consider high Reynolds number flows past an adiabatic flat plate. Whereas this simplification reduces the complexity of the problem, many features of this analysis are applicable to other configurations such as flows past a cone by a straightforward extension of the qualitative results. The viscosity of the fluid is assumed to obey the Chapman relation, the Prandtl number is assumed to be unity, and real gas effects are neglected. For the mean velocity, an analytic expression is developed to approximate the usual Blasius profile. With these simplifications, the problem can be treated analytically by means of the well-known WKB method.

Received Oct. 18, 1993; revision received March 18, 1994; accepted for publication March 9, 1994; presented as Paper 93-2981 at the AIAA 24th Fluid Dynamics Conference, Orlando, FL, July 6–9, 1994.

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II. Formulation

A. Governing Equations

We consider the flow of a parallel, zero pressure gradient hypersonic boundary layer over a flat plate. The surface is assumed to be adiabatic, the fluid is assumed to be a model fluid, i.e., Prandtl number of unity, and the viscosity is assumed to be proportional to the absolute temperature, $\mu/\mu_\infty = T/T_\infty$. The region of interest is considered to be far away from the leading-edge, such that the flow is assumed to be locally parallel, with no shock present. The disturbance is assumed to be an inviscid, two-dimensional normal mode.

For the linear stability problem for a compressible boundary layer over an adiabatic flat plate, the governing nondimensional equation is (see Mack[3])

$$\frac{d^2\hat{p}}{dy^2} - \frac{2}{\bar{M}}\frac{d\bar{M}}{dy}\frac{d\hat{p}}{dy} - (1-\bar{M}^2)\alpha^2\hat{p} = 0 \tag{1}$$

where

$$\bar{M}^2 = \rho(U-c)^2 M_\infty^2$$

where α is the wave number in the freestream direction and is taken to be real in this analysis, and c is the wave speed. In general, c is complex and determines the temporal instability of the perturbation; ρ and U are the mean flow density and velocity profiles, respectively, and M_∞ is the Mach number of the freestream. All of the properties are nondimensionalized with respect to the freestream values.

The variable $\bar{M}$ is referred to as the relative Mach number. It is the local Mach number of the mean flow in the direction of the wave number α, relative to the phase velocity, or conversely $\bar{M}$ can be viewed as the Mach number of the disturbance relative to the sound speed of the mean flow. If c is complex, then $\bar{M}$ is complex. In this investigation, we consider real values of c only, i.e., a neutral disturbance; hence $\bar{M}$ is real only. The formulation of the governing differential equation in terms of the relative Mach number provides a convenient parameter to work with since it contains all of the physical parameters of the flow.

In Eq. (1), the existence of the parameter c will change the character of the equation as the value of c varies. The value of this wave speed, as suggested by Lees and Lin, classifies the neutral wave as either subsonic, for which $1 - 1/M_\infty < c < 1 + 1/M_\infty$; sonic, for which $c = 1 \pm 1/M_\infty$; and supersonic, for which $c < 1 - 1/M_\infty$. Since both the density and velocity profiles vary throughout the boundary layer and are always less than one (except at the edge of the boundary layer), the supersonic case represents the situation where $\bar{M}^2 > 1$ over some portion of the boundary layer, and a multiplicity of solutions can occur, as first noted by Mack.

B. Method of Analysis

Recently Eq. (1) has been analyzed in the high Mach number limit by utilizing the Howarth-Dorodnitsyn change of variable (Smith and Brown, Blackaby et al., and Grubin and Trigub) and also by applying the triple-deck method (Cowley and Hall). In the present study, we shall approach this equation directly by means of the WKB method. The results are then applicable over a wide range of freestream Mach numbers.

Equation (1) can be rewritten as

$$\frac{1}{\bar{M}}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{1}{\bar{M}}\frac{\mathrm{d}\hat{p}}{\mathrm{d}y}\right) - \alpha^2\left(\frac{1-\bar{M}^2}{\bar{M}^4}\right)\hat{p} = 0 \tag{2}$$

Introducing the transformation,

$$\eta = \int \bar{M}^2\,\mathrm{d}y = \int \frac{\bar{M}^2}{\mathrm{d}\bar{M}/\mathrm{d}y}\,\mathrm{d}\bar{M} \tag{3}$$

Eq. (2) can then be reduced to the canonical form

$$\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}\eta^2} - \alpha^2 Q(\eta)\hat{p} = 0 \tag{4}$$

where

$$Q(\eta) = \frac{[1-\bar{M}(\eta)^2]}{\bar{M}(\eta)^4} \tag{5}$$

Clearly the relative sonic point ($\bar{M}^2 = 1$) corresponds to a turning point of Eq. (4), i.e., where $Q(\eta) = 0$. An approximate solution for Eq. (4) can be obtained by means of the WKB method (e.g., see Bender and Orszag[10]). The WKB approximation yields solutions on each side of the turning point, as well as an approximate solution through the turning point. The solution is given as

$$\hat{p}_I(\eta') = C[Q(\eta')]^{-\frac{1}{4}}\exp\left[-\alpha\int_{\eta_{tp}}^{\eta}\sqrt{Q(t)}\,\mathrm{d}t\right] \tag{6}$$

$$\eta' = \eta - \eta_{tp} < 0 \qquad (-\eta') \gg (\alpha)^{-\frac{2}{3}}$$

$$\hat{p}_{II}(\eta') = 2\sqrt{\pi}C\mathrm{Ai}\left(\alpha^{-\frac{2}{3}}\eta'\right) \qquad |\eta'| \ll 1 \tag{7}$$

$$\hat{p}_{III}(\eta') = C[-Q(\eta')]^{-\frac{1}{4}}\sin\left[\alpha\int_{\eta}^{\eta_{tp}}\sqrt{-Q(t)}\,\mathrm{d}t + \frac{\pi}{4}\right] \tag{8}$$

$$\eta' = \eta - \eta_{tp} > 0 \qquad \eta' \gg (\alpha)^{\frac{2}{3}}$$

where C is a constant to be determined, and η_{tp} is the turning point. The variable $\hat{p}_I$ corresponds to the solution in the relative sonic portion of the boundary layer, $\hat{p}_{II}$ that near the turning point, and $\hat{p}_{III}$ that in the relative supersonic region. With the stability problem formulated in this fashion, a single differential equation governs the behavior of the pressure disturbance over the entire boundary layer. Once a relative Mach number distribution is established for the problem at hand, the solution is reduced to an evaluation of the integrals $\int\sqrt{Q(t)}\,dt$ and $\int\sqrt{-Q(t)}\,dt$ over the relative subsonic and supersonic regions, respectively. It should be noted that these equations can be reduced to a uniformly valid approximation by means of the Langer transformation (see Bender and Orszag). This single equation is valid over the entire boundary layer and is given by

$$\hat{p}_{\mathrm{unif}}(\eta') = 2\sqrt{\pi}C\left(\frac{3\alpha}{2}S_0\right)^{\frac{1}{6}}[Q(\eta')]^{-\frac{1}{4}}\mathrm{Ai}\left[\left(\frac{3\alpha}{2}S_0\right)^{\frac{2}{3}}\right] \tag{9}$$

where

$$S_0 = \int\sqrt{Q(\eta)}\,\mathrm{d}\eta$$

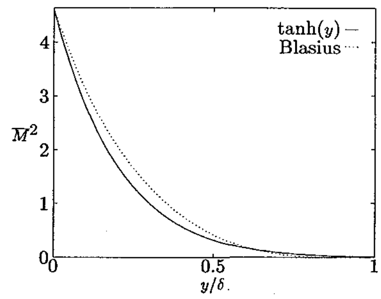

Fig. 1 Relative Mach number profiles for Blasius and tanh($\hat{y}$) velocity profiles with $c = 1$.

Equation (9) was also obtained for this problem by Nayfeh,[11] who utilized a different transformation of independent variable and applied the WKB approximation and the Langer transformation to obtain the single uniformly valid expansion.

Note that this solution retains the multimode character first noted by Mack. In the supersonic region, the solution has a sinusoidal form [given by Eq. (8)], and hence there are mutiple values of the argument of the sine term that will satisfy the boundary conditions {e.g., to satisfy the boundary condition $\hat{p} = 0$, the sine term can only have the argument $[(2n-1)/2]\pi$}. Hence the eigenvalue relation can be shown to be

$$\alpha_n = \left\{\frac{[(2n-1)/2]\pi}{\int_{\eta}^{\eta_{tp}}\sqrt{-Q(t)}\,\mathrm{d}t + (\pi/4)}\right\} \qquad n = (1,2,3,\ldots) \tag{10}$$

III. Solutions for Model Flows

The mean velocity is usually assumed to be the well known Blasius profile. To develop analytical results for the problem, one approximates this profile by

$$U = \tanh(\hat{y}) \tag{11}$$

where $\hat{y} = 3y$.

Figure 1 shows the difference in relative Mach number profile based on the Blasius profile with that based on the approximate profile for $c = 1$. Clearly they have the same qualitative features, and thus Eq. (11) represents a reasonable approximation for the Blasius profile in establishing the relative Mach number distribution.

For a Prandtl number of unity and viscosity that is proportional to the absolute temperature, the density is given by

$$\rho = \left[1 + \frac{(\gamma-1)}{2}M^2(1-U^2)\right]^{-1} \tag{12}$$

These simplified profiles are used to obtain solutions for various values of the wave speed c.

A. Waves with $c = 0$

We first examine the case for $c = 0$, which would correspond to a standing, neutral, two-dimensional wave. Whereas this type of wave is typically highly damped in a hypersonic boundary layer, the behavior of the neutral portion of the wave is examined to demonstrate the procedure in which the eigenfunction and eigenvalue relation are determined using this formulation.

For the case $c = 0$, Eqs. (11) and (12) yield

$$\bar{M} = \frac{M_\infty[\tanh(\hat{y})]}{\sqrt{1 + (\gamma-1)/2M_\infty^2[1-\tanh(\hat{y})^2]}} \tag{13}$$

and thus

$$\frac{d\bar{M}}{dy} = \frac{1}{\sqrt{\Gamma}}\frac{1}{\sqrt{1+\epsilon}}\left(1 - \frac{\bar{M}^2}{M_\infty^2}\right)\left(1 + \frac{\gamma-1}{2}\bar{M}^2\right)^{\frac{1}{2}} \tag{14}$$

where

$$\Gamma = \frac{(\gamma - 1)}{2}$$

$$\epsilon = \frac{2}{(\gamma - 1)M_\infty^2} \tag{15}$$

With this wave speed, the sonic point in the relative Mach number profile given by Eq. (13) [and corresponding to the turning point of Eq. (4)] is near the wall, with the relative supersonic region above the turning point and extending into the freestream. For this case, the appropriate boundary conditions for Eq. (1) are $\hat{p} = 0$ at $y = 0$, and $\hat{p}$ is bounded as $y \to \infty$.

Using Eqs. (13) and (14), the transformation given by Eq. (3) is

$$\eta = \left(\frac{M_\infty^3\sqrt{\Gamma}\sqrt{1+\epsilon}}{2\sqrt{1+\Gamma M_\infty^2}}\right) \times \left\{\sinh^{-1}\left[\frac{\bar{M}\sqrt{\Gamma}}{|1+(\bar{M}/M_\infty)|} - \frac{\sqrt{\Gamma}}{\Gamma|M_\infty + \bar{M}|}\right] + \sinh^{-1}\left[\frac{\bar{M}\sqrt{\Gamma}}{|1-(\bar{M}/M_\infty)|} + \frac{\sqrt{\Gamma}}{\Gamma|M_\infty - \bar{M}|}\right]\right\} - M_\infty^2\sqrt{1+\epsilon}\sinh^{-1}(\bar{M}\sqrt{\Gamma}) \tag{16}$$

To apply the WKB approximations given by Eqs. (6–8), one must evaluate the integrals $\int\sqrt{-Q(t)}\,\mathrm{d}t$ and $\int\sqrt{Q(t)}\,\mathrm{d}t$. For the relative supersonic region, the integral is written as

$$S_0 = \int_{\eta(\bar{M}=1)}^{\eta} \frac{\sqrt{\bar{M}^2 - 1}}{\bar{M}^2}\,\mathrm{d}\eta$$

Using Eqs. (3), (13), and (14), one can reduce this integral to

$$S_0(y,\ M_\infty) = \sqrt{\Gamma}\sqrt{1+\epsilon}\int_{\bar{M}=1}^{\bar{M}(y,M_\infty)} \frac{\sqrt{\bar{M}^2-1}}{\left[1-(\bar{M}^2/M_\infty^2)\right]\sqrt{1+\Gamma\bar{M}^2}}\,\mathrm{d}\bar{M}$$

which yields

$$S_0(y,\ M_\infty) = \left(\frac{M_\infty^2\sqrt{\Gamma}\sqrt{1+\epsilon}}{2\sqrt{1+\Gamma M_\infty^2}}\right) \times \left(\sinh^{-1}\left\{\frac{\bar{M}(y,M_\infty)\sqrt{\Gamma}}{|1-[\bar{M}(y,M_\infty)/M_\infty]|} + \frac{\sqrt{\Gamma}}{\Gamma|M_\infty - \bar{M}(y,M_\infty)|}\right\} - \sinh^{-1}\left\{\frac{\bar{M}(y,M_\infty)\sqrt{\Gamma}}{|1+[\bar{M}(y,M_\infty)/M_\infty]|} - \frac{\sqrt{\Gamma}}{\Gamma|M_\infty + \bar{M}(y,M_\infty)|}\right\}\right) \tag{17}$$

A similar method is employed to obtain the integral in the relative subsonic region, below the turning point. The eigenfunction given by Eq. (9) can be now be written

$$\hat{p}_{\text{unif}}(y,\ M_\infty) = 2\sqrt{\pi}C\left[\frac{3\alpha}{2}S_0(y,\ M_\infty)\right]^{\frac{1}{6}} \times [Q(y,\ M_\infty)]^{-\frac{1}{4}}\mathrm{Ai}\left\{\left[\frac{3\alpha}{2}S_0(y,\ M_\infty)\right]^{\frac{2}{3}}\right\} \tag{18}$$

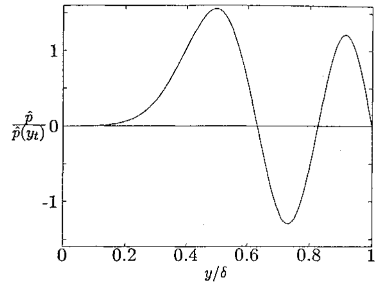

Fig. 2 Pressure perturbation for M_∞ = 6, n = 3, and c = 0.

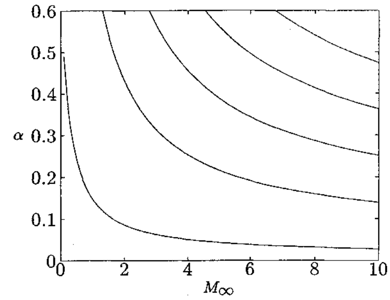

Fig. 3 Eigenvalue relation for c = 0.

where $S_0(y, M_\infty)$ is given by Eq. (17) for the relative supersonic region, and a similar relation is obtained for the subsonic portion, using $Q(y, M_\infty)$, given by Eq. (5) with $\bar{M}^2 < 1$.

Figure 2 shows the third mode of the eigenfunction for a freestream Mach number of 6. This solution has the expected behavior, as inferred from the relative Mach number profile for this case with the disturbance decaying to zero in the relative subsonic region near the wall, attaining an oscillatory nature in the supersonic region. The amplitude of the disturbance eventually levels to a constant value in the freestream outside the boundary layer.

To obtain the eigenvalue relation given by Eq. (10), Eq. (17) is evaluated at the limits of integration, i.e., between the turning point and the edge of the boundary layer, which yields

$$\alpha_n = \frac{(2n-1)\pi}{2}\left(\left(\frac{M_\infty^2\sqrt{\Gamma}\sqrt{1+\epsilon}}{2\sqrt{1+\Gamma M_\infty^2}}\right) \times \left\{\sinh^{-1}\left[\frac{\bar{M}_\delta\sqrt{\Gamma}}{|1-(\bar{M}_\delta/M_\infty)|} + \frac{\sqrt{\Gamma}}{\Gamma|M_\infty - \bar{M}_\delta|}\right] - \sinh^{-1}\left[\frac{\bar{M}_\delta\sqrt{\Gamma}}{|1+(\bar{M}_\delta/M_\infty)|} - \frac{\sqrt{\Gamma}}{\Gamma|M_\infty + \bar{M}_\delta|}\right] - \sinh^{-1}\left[\frac{\sqrt{\Gamma}}{1-(1/M_\infty)} + \frac{1}{\sqrt{\Gamma}(M_\infty - 1)}\right] + \sinh^{-1}\left[\frac{\sqrt{\Gamma}}{1+(1/M_\infty)} - \frac{1}{\sqrt{\Gamma}(M_\infty + 1)}\right]\right\} + \pi/4\right)^{-1} \tag{19}$$

Equation (19) is shown in Fig. 3 (using $\gamma = 1.4$) for the first five neutral modes and clearly demonstrates the multiplicity of the solutions. The solutions for the eigenvalue and the eigenfunction exhibit all of the features expected for this wave speed, as inferred from the relative Mach number distribution. However, in this formulation these approximate solutions were obtained analytically and are valid for a wide range of freestream Mach numbers.

B. Waves with $c = 1$

We now apply the same procedure to the case of the disturbance with a wave speed equal to the mean velocity at the edge of the boundary layer. As discussed by Mack, this wave is the most significant of the noninflectional neutral modes, which have phase velocities in the range $1 \leq c \leq 1 + 1/M_\infty$. This mode, with a wave speed of $c = 1$, was examined in the limit $M \to \infty$ by Smith and Brown

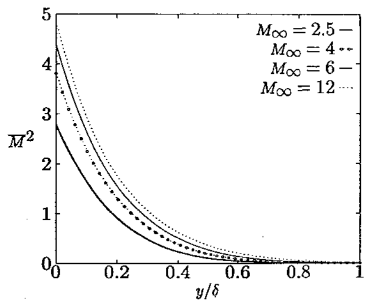

Fig. 4 Relative Mach number profiles for several freestream Mach numbers with $c = 1$.

and Cowley and Hall and was treated numerically by Mack. In this situation, the turning point is at the edge of the boundary layer, and the relative supersonic region appears near the wall. Here, in contrast to the previous case, there is the possibility that waves may be trapped in the relative supersonic region, between the wall and the turning point. This case is also of importance since this neutral wave is accompanied by a family of unstable waves with $c < 1$, and these subsonic waves (by the Lees and Lin criteria) are considered to be the most unstable ones.

Using Eqs. (11) and (12), one can find that the expression for the relative Mach number for the case $c = 1$ is

$$\bar{M} = \frac{M_\infty[\tanh(\hat{y}) - 1]}{\sqrt{1 + (\gamma - 1/2)M_\infty^2[(1 - \tanh(\hat{y})^2]}} \tag{20}$$

This profile is shown in Fig. 4 for several freestream Mach numbers. For this case, the appropriate boundary conditions for Eq. (1) are

$$\frac{\mathrm{d}\hat{p}(0)}{\mathrm{d}y} = 0, \qquad \hat{p}(\delta) = 0$$

With this expression, Eq. (3) yields

$$\eta = M_\infty^2 \int \left[\frac{(z-1)^2}{1 + (1/\epsilon)(1 - z^2)}\right] \frac{\mathrm{d}z}{(1 - z^2)} \tag{21}$$

where

$$z = \tanh(\hat{y})$$

The expression for $\eta = \eta(z, M_\infty)$ is now

$$\eta(z, M_\infty) = M_\infty^2 \left\{ \frac{1 + \left(1/2\Gamma M_\infty^2\right)}{\sqrt{1 + \left(1/\Gamma M_\infty^2\right)}} \times \log\left[\frac{\sqrt{1 + \left(1/\Gamma M_\infty^2\right)} - z}{\sqrt{1 + \left(1/\Gamma M_\infty^2\right)} + z}\right] - \log\left[1 + \left(1/\Gamma M_\infty^2\right) - z^2\right] + 2\log(z+1) \right\} \tag{22}$$

To develop the eigenvalue relation and the associated eigenfunction for the pressure perturbation, one follows the same procedure as in the previous case, i.e., an evaluation of the integral

$$S_0 = \int_{\eta_{y=0}}^{\eta_{y=yt}} \frac{\sqrt{\bar{M}^2 - 1}}{\bar{M}^2}\,\mathrm{d}\eta \tag{23}$$

in the supersonic region near the wall below the turning point and

$$S_0 = \int_{\eta_{y=yt}}^{\eta_{y=\delta}} \frac{\sqrt{1 - \bar{M}^2}}{\bar{M}^2}\,\mathrm{d}\eta \tag{24}$$

for the region above the turning point, where $\bar{M}$ is now given by Eq. (20). Utilizing the transformation given by Eq. (21), one can now give Eq. (23) by

$$S_0\left(z, M_\infty^2\right) = \int \sqrt{\frac{M_\infty^2(z-1)^2}{1 + (1/\epsilon)(1 - z^2)} - 1}\,\frac{\mathrm{d}z}{(1 - z^2)}$$

which, after some algebraic manipulation, may be written as

$$S_0\left(z, M_\infty^2\right) = \int \frac{\mathrm{d}z}{\sqrt{(a-z)(z-b)(z-c)(z-d)}} + \int \frac{A\,\mathrm{d}z}{(z+1)\sqrt{(a-z)(z-b)(z-c)(z-d)}} - \int \frac{B\,\mathrm{d}z}{(z-1)\sqrt{(a-z)(z-b)(z-c)(z-d)}} \tag{25}$$

where

$$a = \sqrt{1 + \frac{1}{\Gamma M_\infty^2}}$$

$$b = \frac{\sqrt{\Gamma\left[1 + \left(1/M_\infty^2\right)\right] + \left(1/M_\infty^2\right)} - 1}{1 + \Gamma}$$

$$c = -\sqrt{1 + \frac{1}{\Gamma M_\infty^2}}$$

$$d = \frac{-\sqrt{\Gamma\left[1 + \left(1/M_\infty^2\right)\right] + \left(1/M_\infty^2\right)} - 1}{1 + \Gamma}$$

$$A = \tfrac{1}{2}[1 + bd - (b + d)]$$

$$B = -\tfrac{1}{2}[1 + bd + (b + d)]$$

The first integral in Eq. (25) is easily recognized as an elliptic integral; the integrands in the second and third integrals may be reduced to Jacobian elliptic functions, and the integration will yield (see Byrd and Freidman[12])

$$S_0\left(z, M_\infty^2\right) = gF(\phi, k) + \frac{Ag}{\lambda_A^2(b+1)}\left[\left(\lambda_A^2 - \lambda^2\right)\Pi\left(\phi, \lambda_A^2, k\right) + \lambda^2 F(\phi, k)\right] - \frac{Bg}{\lambda_B^2(b-1)}\left[\left(\lambda_B^2 - \lambda^2\right)\Pi\left(\phi, \lambda_B^2, k\right) + \lambda^2 F(\phi, k)\right] \tag{26}$$

where $F(\phi, k)$ and $\Pi(\phi, \lambda_3^2, k)$ are incomplete elliptic integrals of the first and third kind, respectively, with

$$g = \frac{2}{\sqrt{(a-c)(b-d)}}$$

$$k^2 = \frac{(a-b)(c-d)}{(a-c)(b-d)}$$

$$\phi = \phi(z) = \sin^{-1}\left[\frac{(a-c)(z-b)}{(a-b)(z-c)}\right]^{\frac{1}{2}}$$

$$\lambda_A^2 = \frac{(1+c)(a-b)}{(1+b)(a-c)}$$

$$\lambda_B^2 = \frac{(1-c)(a-b)}{(1-b)(a-c)}$$

$$\lambda^2 = \frac{a-b}{a-c}$$

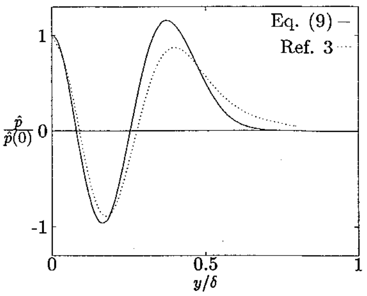

Fig. 5 Pressure perturbation compared with results of Mack for M_∞ = 10, n = 3, and c = 1.

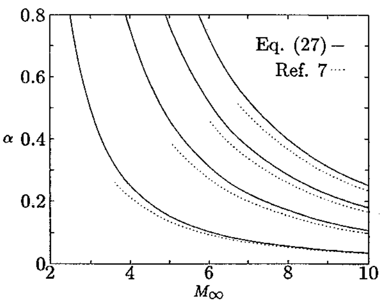

Fig. 6 Eigenvalue relation for c = 1 compared with the numerical results of Cowley and Hall.

A similar relation is developed for the relative subsonic portion of the boundary layer. These integrals are then substituted into Eqs. (9) and (10) to obtain the eigenfunction and eigenvalue relations, as in the previous case.

Figure 5 shows the eigenfunction given by Eq. (9) corresponding to the third mode of the eigenvalue relation. Note that the wave shows a decay as it passes through the turning point and decays to zero in the sonic portion, as shown by previous researchers. Hence this result accurately portrays the situation when the wave essentially is trapped between the wall and the sonic line within the boundary layer and may be accompanied by a family of unstable waves. Figure 5 also shows the direct numerical solution of Eq. (1), obtained by Mack for the same case. The largest discrepancy between the present result and that of Mack appears near the turning point. In Mack's computation, a Blasius type profile is used for the mean velocity; as shown previously in Fig. 1, the largest deviation in relative Mach number profiles using a Blasius velocity profile and a tanh velocity profile occurs near the turning point. Hence, the variation of the two results is due in large part to the minor difference in the mean profiles used in each study. However, there is still a good qualitative comparison between the results.

The eigenvalue relation is expressed as a function of Mach number only for each mode by evaluating these integrals given by Eq. (23) between the wall and the sonic point, giving

$$\alpha_n = \frac{[(2n-1)/2]\pi}{S_0(z_t, M_\infty) - S_0(0, M_\infty)} \tag{27}$$

This relation is shown in Fig. 6. Referring back to Fig. 4, it is interesting to note that as the freestream Mach number increases, even with the critical point ($\bar{M}^2 = 0$) at a constant position, namely, at the edge of the boundary layer, the turning point of this problem moves out towards the edge of the boundary layer. For the solution to satisfy the boundary conditions both at the wall and the turning point, the wavelength of the disturbance must increase. The eigenvalue relation obtained, shown in Fig. 6, reflects this trend, with the wave number monotonically decreasing with increasing Mach number. Also shown in Fig. 6 are the direct numerical results of Cowley and Hall for the case $c = 1$, which show good qualitative comparison with the results obtained here, especially at large values of the freestream Mach numbers.

A high Mach number approximation for the eigenvalue relation is obtained in this formulation by expanding Eq. (26) by use of the series representation of the incomplete elliptic integrals (see Byrd and Freidman) and retaining the first few terms in the series, i.e.,

$$F(\phi, k) \approx \phi + \frac{3k^2}{8}(\phi - \sin\phi\cos\phi) - \frac{15k^4}{128}[3\phi - (3 + \sin^2\phi)\sin\phi\cos\phi] + \cdots \tag{28}$$

$$\Pi(\phi, \lambda_3^2, k) \approx \phi - \frac{3k^2}{4\lambda_3^2}\left[\phi + \frac{\lambda_3^2}{4}(\phi - \sin\phi\cos\phi)\right] + \cdots \tag{29}$$

Table 1 Comparison of the eigenvalues αM_∞^2 with the results of Smith and Brown and Cowley and Hall

n	Smith and Brown	Cowley and Hall	Eq. (30)
1	1.757	3.401	3.427
2	8.785	9.786	10.28
3	15.81	16.50	17.13
4	22.84	23.36	23.98
5	29.87	30.28	30.84

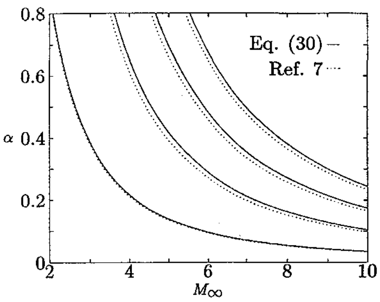

Fig. 7 Asymptotic eigenvalue relation given by Eq. (30) compared with asymptotic results of Cowley and Hall.

Using the series representations given by Eqs. (28) and (29) and substituting into Eq. (26) for $S_0(z, M_\infty^2)$, after considerable algebraic manipulation an expression for large Mach numbers for the eigenvalue relation given by Eq. (10), given by

$$\alpha = \frac{3\pi}{2}\frac{(2n-1)(1+\Gamma)}{\sqrt{\Gamma}a_0(\Gamma)M_\infty^2}\left[1 + \frac{a_1(\Gamma)}{a_0(\Gamma)M_\infty^2} + \cdots\right]^{-1} \tag{30}$$

where

$$a_0(\Gamma) \approx \frac{4[\sqrt{1+\Gamma}(3 - 4\sqrt{\Gamma})]}{1 + \sqrt{\Gamma}}$$

$$a_1(\Gamma) \approx 2\sqrt{1+\Gamma}(9.5\Gamma + 4.5\sqrt{\Gamma} - 3.5)$$

which in the limit $M_\infty \to \infty$ is further reduced to

$$\alpha \approx \frac{3\pi}{2}\frac{(2n-1)(1+\Gamma)}{\sqrt{\Gamma}a_0(\Gamma)M_\infty^2} \tag{31}$$

This asymptotic expression for the eigenvalue relation with $\gamma = 1.4$ and $n = 1$–4 is shown in Fig. 7 compared with the eigenvalues given by Cowley and Hall, obtained by a direct numerical solution of Eq. (1) for a Blasius boundary layer. These results for the eigenvalues αM_∞^2 of the acoustic modes are also shown in Table 1, along with the asymptotic results of Smith and Brown. The prediction in this formulation, even in the limit of high Mach numbers, compares very favorably with the numerical and asymptotic results.

IV. Summary and Conclusions

We have shown that the compressible form of the Rayleigh equation for two-dimensional inviscid disturbances can be formulated in such a fashion that the WKB method may be utilized throughout the entire boundary layer. This method of analyzing the linear two-dimensional stability problem provides a uniformly valid approximate solution for the disturbance and for the associated eigenvalue relation. The solutions obtained by using this method have shown consistent qualitative agreement with the numerical computations of Mack and Cowley and Hall, as well as the asymptotic results of Smith and Brown. This was shown for the cases of $c = 0$ and 1, the latter case representing the situation when the relative sonic point is near the edge of the boundary layer, and the pertubations are considered to be all acoustic modes.

In the present study, the governing equation is derived such that the perturbation is expressed in terms of the relative Mach number, a parameter that contains all of the flowfield parameters. As a result, the only values that need to be calculated to obtain the eigenvalues and corresponding eigenfunctions are the integrals $\int \sqrt{-Q(t)}\, dt$ and $\int \sqrt{Q(t)}\, dt$ that are functions of the relative Mach number distribution only. Hence, this method could be applied to a variety of different problems, as long as the relative Mach number distribution could be determined for the problem, e.g., a flat plate with heat transfer. Of course, for certain distributions, these integrals may not be evaluated in closed form; then either an approximate integral may be obtained or the integration may be performed numerically. Thus, this approach provides a much simpler method than a full numerical integration of the governing equation.

Acknowledgment

The authors would like to acknowledge the contribution of Jim Sonnenmeier of the Department of Mechanical and Aerospace Engineering, whose help was invaluable in the preparation of this paper and is greatly appreciated.

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